The art of counting infinity (Part 2)

by Rudolf Taschner

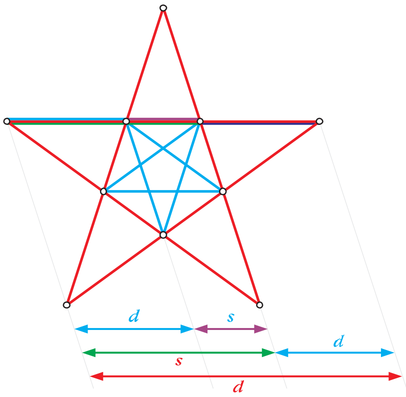

Fig. 2: The pentagram

Continuing from part 1 of this article published in volume 13 Number 2…

The pentagram was used by the Pythagoreans as the seal of their secret society and according to a quite plausible legend here again they tried to discover the common measure of two lines. The pentagram, drawn in red, distinguishes itself by the remarkable fact that the points of intersection of the diagonals are the vertices of another, smaller, inscribed pentagram, drawn in blue. Thus we see not only the lengths of the red diagonals d and the red laterals s, the connection between two contiguous points, but also the lengths of the blue diagonals d and the blue laterals s. How many times will the red lateral fit into the red diagonal? One glance at the symmetry of the figure will tell us: once, with the blue diagonal as remainder. Now we ask, taking our cue from the ancient Greeks' method of alternate removal, how many times will the blue diagonal fit into the red lateral? Again, the symmetry of the figure tells us: once, the blue lateral being the remainder. And yet again, according to the method of alternate removal, we have to ask ourselves, how many times will the blue lateral fit into the blue diagonal? But we have already asked this very question concerning the red lateral and the red diagonal. It will always lead to the same series of answers, a never-ending sequence, just as in every pentagram yet another pentagram can be inscribed, on and on to infinity.

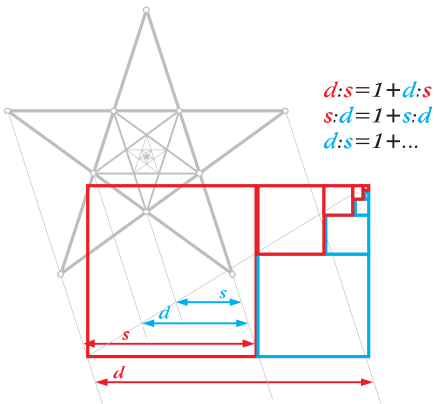

Fig. 3: The division of the diagonal by the lateral of the pentagram

The Pythagoreans formulated their discovery by saying, "The diagonal and the lateral of a pentagram have no common measure". No straight line, however conceivably short, will fit in both the diagonal and the lateral of a pentagram as an integer. The ratio of diagonal and lateral of a pentagram has since been fittingly defined as "irrational".

This, however, is only the beginning. Archimedes had already discovered another geometrical ratio which later turned out to be "irrational": the ratio between circumference and diameter of a circle, which is designated by the Greek letter π. Since it is irrational, π cannot be represented as a finite decimal number - the commonly known value of 3.14159 is only a crude approximation - and cannot ever have a periodical sequence in its decimals. At around 1600, the Dutchman Ludolph van Ceulen inscribed - only as a mental operation of course - a polygon of 4.6 quadrillion vertices into a circle, and thus computed 35 digits after the decimal point of π. Naturally, this is not the exact value either. The most powerful computer available today has produced a far greater number of digits after the decimal point; the current record is at more than 1.24 trillion digits. What this means is best conveyed by the image of a book, or rather a huge library with a quarter of a million books containing a thousand pages each, 248 million pages with 5000 characters a page, and page after page most monotonously filled with the decimal digits of π, and no hope of even the slightest regularity in the sequence of numbers. On the contrary, the figures 0 to 9 seem to crop up as randomly as the numbers 0 to 36 at the roulette tables of Monte Carlo.

The weirdest thing about this obstinate computation of the digits of π is that even knowing a trillion digits will tell us next to nothing about its further decimal development. Whatever the might and capacity of the most powerful computer ever to be devised by man, it will never be able to deliver to us the infinite number of decimal digits of π.

And this is only the starting point of the story of infinity, because both the irrational ratio of diagonal and lateral in a pentagram and the irrational ratio of circumference and diameter in a circle that we call π, are only two out of an infinite number of irrational ratios, which can be visualised as points on a scale - even though we should mistrust plausible arguments drawn from sensory perceptions, particularly when dealing with matters infinite. We can also accommodate all "rational" ratios on this scale, the results of divisions of integers. The German mathematician Georg Cantor even demonstrated that these rational ratios can be enumerated as completely as the numbers themselves. In other words, there is a procedure that keeps producing rational ratios and will, assuming one waits around long enough, include each arbitrary given rational ratio. This may not seem too amazing. What Cantor went on to prove, though, was astounding indeed. If somebody came up with a procedure that kept producing irrational ratios such as the two mentioned before, it would *never* be able to be exhaustive, no matter how sophisticated it might be. There would always be one more irrational ratio than one could track down and which the procedure had failed to discover.

Cantor thought that, beyond the infinity of numbers he had discovered, there existed even more complex degrees of infinity, saying that the infinity of points on a scale by far surpasses the infinity of numbers. This deduction, however, might possibly take us the wrong way. The eminent mathematician Luitzen Egbertus Jan Brouwer even had serious doubts as to whether human logic was permitted to ponder the infinite in the same way it does the world of the finite. In this case, we would not be free to conceive and represent infinity as a given entity and object of mathematical enquiry, unlike numbers. Rather, it is a borderland term which eludes the human quest for knowledge.

Therefore we have but the poet's words to speak about infinity. There is hardly a more poignant description of the perception of infinity than this key passage from Robert Musil's novel "The Confusions of Young Törless". To overcome some inner restlessness, Törless lies down on the lawn of the boarding school. He "threw himself down in

Infinity

the pale, rustling grass at the foot of the almost windowless side wall. The sky spread out above him, in that pale, ailing blue, so typical of autumn, and little white round clouds scudded across it. Törless lay stretched out on his back and, dreaming vaguely, squinted between two treetops in front of him that were shedding their leaves. (…)

And suddenly he noticed– and he felt as though this was happening for the first time – how high the sky really was.

It came to him like a shock. Right above him there gleamed a little blue, unimaginably deep hole between the clouds.

It seemed to him that if one had a long, long ladder, one should be able to climb into that hole. But the further he pushed his way in, lifting himself up with this gaze, the further away the blue glowing background retreated. And yet he felt as though it should be possible to reach it and hold it, merely with one's gaze. The desire became painfully intense.

It was as though the power of vision, strained to its limit, was flinging glances like arrows between the clouds, and as though, aim as far as they might, they always fell short.

Törless thought about this now; he tried to remain as calm and sensible as he possibly could. 'Of course there is no end,' he said to himself, 'it goes on and on, ever onward, into infinity.' He kept his eyes fixed on the sky and said this out loud, as though to test the power of a magic spell. But without success; the words said nothing, or rather they said something quite different, as though they were referring to the same object, but to another strange, indifferent side of it.

'Infinity!' Törless knew the word from maths class. He had never imagined anything particular by it. It was forever returning; someone must have invented it once, and since then it had become possible to calculate with it as surely as one did with anything solid. It was whatever its value happened to be in the calculation; Törless had never ventured further than that.

And now, all of a sudden, the idea flashed through him that there was something terribly unsettling about the word. It struck him as a concept that had formerly been tamed, one with which he had performed his daily little tricks, and which had now been suddenly unleashed. Something beyond understanding, something wild and destructive seemed to have been put to sleep by the work of some clever inventor, and had now suddenly been woken to life, and grown terrible before him. There, in that sky, it now stood vividly above him and menaced and mocked.

Finally he closed his eyes, because the vision tormented him so."

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